Hybrid iterative-direct domain decomposition based solvers for the time-harmonic Maxwell equations

1st workshop of the joint INRIA-UIUC laboratory for Petascale Computing

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Opening: NACHOS project-team

Scientific objectives

- Design, analysis and validation of numerical methods and high performance resolution algorithms for the computer simulation of evolution problems in complex domains and heterogeneous media
- Research directions
  - High-order finite element discretization methods on simplicial meshes
  - Hybrid explicit/implicit time integration strategies
  - Domain decomposition resolution algorithms
  - High performance computing related aspects

Computational electromagnetics

- System of Maxwell equations
- Interaction of EM fields with biological tissues
- Interaction of charged particles with EM fields (Vlasov/Maxwell equations)

Computational geoseismics

- System of elastodynamic equations
- Propagation of seismic waves in heterogeneous geological media
- Numerical modeling of planar and non-planar faults (earthquake dynamics)

Opening: context and goal

- Modeling issues
  - Electromagnetic wave propagation in heterogeneous media
  - Time-harmonic regime, high frequency (100 MHz to 3 GHz)
  - Irregularly shaped domains, complex geometrical features
- Target applications
  - Human exposure to electromagnetic fields
  - Medical applications (in-body miniaturized sensor/antenna design for wireless monitoring systems)
- Numerical ingredients
  - Unstructured meshes (triangles in 2D, tetrahedra in 3D)
  - High order Discontinuous Galerkin DG discretization methods
- Goal of this study
  - Design of domain decomposition based hybrid iterative-direct solvers for algebraic systems resulting from DG discretization
  - Part of this work is undertaken in the context of the PhyLeaS associate team

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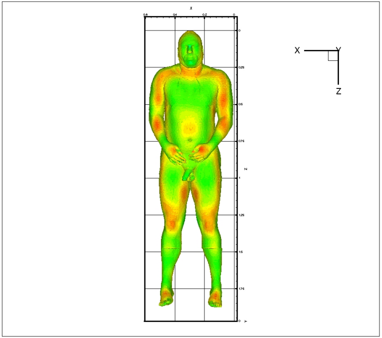
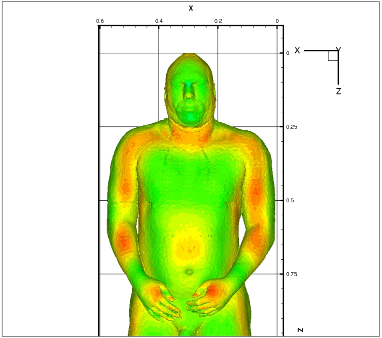
Opening: example application

- Human exposure to electromagnetic fields
  - Multi-parametric studies, uncertainty quantification (source position, morphology, electromagnetic parameters)





- Plane wave exposure (F=2.14 GHz)
- Tetrahedral mesh: 899,872 vertices and 5,335,521 elements
- Discretization by a DG-$\mathbb{P}_2$ method: 320,131,260 d.o.f

Opening: related works

- Full set of Maxwell equations
- High order finite element discretization
- Domain decomposition solver
  - J.J.W. van der Vegt, M.A. Botchev *at al.*, University of Twente
    - High order edge element, *hp*-adaptivity, sparse direct solver
  - P. Solin, `h-FEM group`, University of Nevada
    - High order edge element, *hp*-adaptivity, sparse direct solver
  - J. Zou *at al.*, The Chinese University of Hong Kong
    - Low order edge element, non-overlapping DD solver
  - J.-F. Lee *at al.*, Ohio State University
    - High order edge element, non-overlapping DD solver, non-matching grid
    - Schwarz algorithm with Robin interface conditions
  - A. Schädle, F. Schmidt *at al.*, ZIB
    - Schwarz algorithm with transparent interface conditions approximated by a PML method
    - High order edge element (mixed triangular/quadrangular mesh)
    - Scattering by periodic structures

Opening: related works

- J. Jin *at al.*, UIUC, Department of Electrical and Computer Engineering
  - Full set of Maxwell equations
  - Time-harmonic and time-domain formulations
  - High order edge element discretization
  - Domain decomposition solver

- Mini-symposium at the 19th International Conference on Domain Decomposition Methods (DD19), August 17-22, 2009, Zhangjiajie of China
  Domain decomposition methods for electromagnetic wave propagation problems
  - J. Jin
  - J.-F. Lee, Ohio State University
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    State Key Laboratory of Millimeter Waves
  - M. Gander, University of Geneva, Mathematics Section

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Outline

The time-harmonic Maxwell equations

$$\begin{cases} \varepsilon \mathbf{i}\omega \mathbf{E} - \mathrm{rot}(\mathbf{H}) & = & 0, \\ \mu \mathbf{i}\omega \mathbf{H} + \mathrm{rot}(\mathbf{E}) & = & 0 \end{cases}$$

- $\mathbf{E} = \mathbf{E}(\mathbf{x})$: electric field
- $\mathbf{H} = \mathbf{H}(\mathbf{x})$: magnetic field
- $\varepsilon = \varepsilon(\mathbf{x})$: electric permittivity
- $\mu = \mu(\mathbf{x})$: magnetic permeability
- $\sigma = \sigma(\mathbf{x})$: electric conductivity
- Vacuum impedance: $z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$
- Boundary conditions
  - PEC boundary (Γ^m): $\mathbf{n} \times \mathbf{E} = 0$
  - Absorbing boundary (Γ^a): $\mathbf{n} \times \mathbf{E} + z\mathbf{n} \times (\mathbf{n} \times \mathbf{H}) = \mathbf{n} \times \mathbf{E}^{\infty} + z\mathbf{n} \times (\mathbf{n} \times \mathbf{H}^{\infty})$

The time-harmonic Maxwell equations

Pseudo-conservative system form

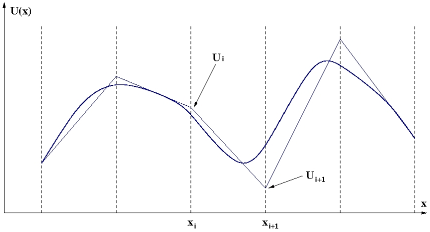
$$QW_t + \nabla \cdot F(\mathbf{W}) = \mathbf{J}$$

$$F_x(\mathbf{W}) = \begin{bmatrix} 0_{3\times3} & N_x \\ -N_x & 0_{3\times3} \end{bmatrix} \mathbf{W} = {}^t(0, H_z, -H_y, 0, -E_z, E_y)$$

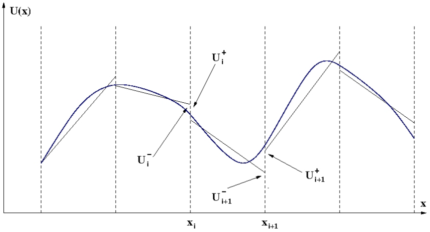
$$F_y(\mathbf{W}) = \begin{bmatrix} 0_{3\times3} & N_y \\ -N_y & 0_{3\times3} \end{bmatrix} \mathbf{W} = {}^t(-H_z, 0, H_x, E_z, 0, -E_x)$$

$$F_z(\mathbf{W}) = \begin{bmatrix} 0_{3\times3} & N_z \\ -N_z & 0_{3\times3} \end{bmatrix} \mathbf{W} = {}^t(H_y, -H_x, 0, -E_y, E_x, 0)$$

$$N_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \;,\; N_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \;,\; N_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



FE, continuous P1 interpolation



DG, local P1 interpolation

- Can easily deal with discontinuous coefficients and solutions
- Can handle unstructured, non-conforming meshes
- High order accurate methods with compact stencils
- Naturally lead to discretization (h-) and interpolation order (p-) adaptivity
- Amenable to efficient parallelization
- But lead to larger problems compared to continuous finite element methods

DG for electromagnetic wave propagation in heterogeneous media

- The basic support of the DG method is the element (triangle in 2D and tetrahedron in 3D)
- Heterogeneity is ideally treated at the element level
    - Mesh generation process is simplified
    - Discontinuities occur at material (i.e element) interfaces
- Wavelength varies with ϵ and μ
    - For a given mesh density, approximation order can be set on an element basis to fit to the local frequency resolution criteria

Discretization of irregularly shaped domains

- Unstructured meshes are preferred
- Local refinement is made easier by allowing non-conformity
- Non-conformity opens the route to the coupling of different discretization methods (e.g structured/unstructured)

- Triangulation: $\mathcal{T}_h = \bigcup_{i=1}^{N} \tau_i$
- Assume $\mathbf{J} = 0$ for simplicity of the presentation
- φ is a sufficiently regular test function
- $\mathbf{n}_{ij} = {}^t(n_{ij}^{x_1}, n_{ij}^{x_2}, n_{ij}^{x_3})$

$$\int_{\tau_i} \varphi \left(\mathbf{i}\omega Q\mathbf{W} + \nabla \cdot F(\mathbf{W}) \right) d\mathbf{x} = 0$$

$$\Leftrightarrow \quad \int_{\tau_i} \mathbf{i}\omega Q\mathbf{W}\varphi d\mathbf{x} - \int_{\tau_i} \nabla\varphi \cdot F(\mathbf{W}) d\mathbf{x} + \int_{\partial\tau_i} \left(F(\mathbf{W}) \cdot \mathbf{n}\right) \varphi d\sigma = 0$$

- Local set of degrees of freedom for τ_i: $\mathbf{W}_{ij} \in \mathbf{R}^6$
- Local approximation: $\mathbf{W}_i(\mathbf{x}) \in \mathcal{P}_i = \mathbb{P}_m[\tau_i]$ and $\mathbf{W}_i(\mathbf{x}) = \sum_{j=1}^{d_i} \mathbf{W}_{ij}\varphi_{ij}(\mathbf{x})$

$$\Leftrightarrow \quad \mathbf{i}\omega Q_i \int_{\tau_i} \mathbf{W}_i \varphi d\mathbf{x} - \int_{\tau_i} \nabla\varphi \cdot F(\mathbf{W})_i d\mathbf{x} + \sum_{j\in\mathcal{V}_i} \int_{a_{ij}} \left(F(\mathbf{W}) \cdot \mathbf{n}_{ij}\right) \varphi d\sigma = 0$$

Discontinuous Galerkin discretization methods

Formulation

$$F(\mathbf{W})_i \equiv F(\mathbf{W})_{|\tau_i} \quad \text{and} \quad Q_i = \begin{bmatrix} \varepsilon_i I_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & \mu_i I_{3\times 3} \end{bmatrix}$$

- $\phi_i = (\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{id_i})$
- $\mathcal{V}_i = \{j | \tau_i \cap \tau_j \neq 0\}$ and $a_{ij} = \tau_i \cap \tau_j$
- Calculation of the boundary term on a_{ij}: numerical flux
  - Centered scheme : $F(\mathbf{W})|_{a_{ij}} \approx \dfrac{F(\mathbf{W})_i + F(\mathbf{W})_j}{2}$
  - Upwind scheme : $F(\mathbf{W})|_{a_{ij}} \approx F^+(\mathbf{W})_i + F^-(\mathbf{W})_j$
- Integration by parts + algebraic manipulations

$$\begin{aligned} \sum_{j\in\mathcal{V}_i} \int_{a_{ij}} (F(\mathbf{W}) \cdot \mathbf{n}_{ij})\, \varphi d\sigma &= \frac{1}{2} \int_{\tau_i} ((\nabla \cdot F(\mathbf{W})_i)\, \varphi + \nabla\varphi \cdot F(\mathbf{W})_i)\, d\mathbf{x} \\ &+ \frac{1}{2} \sum_{j\in\mathcal{V}_i} \int_{a_{ij}} (F(\mathbf{W})_j \cdot \mathbf{n}_{ij})\, \varphi d\sigma \end{aligned}$$

Weak formulation

$$\mathbf{i}\omega Q_i \int_{\tau_i} \mathbf{W}_i \varphi d\mathbf{x} + \frac{1}{2}\int_{\tau_i} \left(\left(\sum_{k=1}^{3} \partial_{x_k} \Psi^{x_k} \mathbf{W}_i \right) \varphi - \sum_{k=1}^{3} \partial_{x_k} \varphi \Psi^{x_k} \mathbf{W}_i \right) d\mathbf{x}$$
$$+ \frac{1}{2} \sum_{j \in \mathcal{V}_i} \int_{a_{ij}} M_{ij} \mathbf{W}_j \varphi d\sigma = 0$$

$$\left\{ \begin{array}{lcl} \Psi^{x_k} & = & \begin{bmatrix} 0_{3\times 3} & N_{x_k} \\ -N_{x_k} & 0_{3\times 3} \end{bmatrix} \\ M_{\mathbf{n}} & = & \sum_{k=1}^{3} n^{x_k} \Psi^{x_k} = \begin{bmatrix} 0_{3\times 3} & N_{\mathbf{n}} \\ -N_{\mathbf{n}} & 0_{3\times 3} \end{bmatrix} \text{ and } M_{ij} = M_{\mathbf{n}_{ij}} \\ N_{\mathbf{n}} & = & \sum_{k=1}^{3} n^{x_k} N_{x_k} \\ N_{ij} & = & N_{\mathbf{n}_{ij}} \end{array} \right.$$

Discontinuous Galerkin discretization methods

Matricial system

Local algebraic system of equations

$$
\begin{aligned}
2\mathbf{i}\omega\mathcal{Y}_i\mathbf{W}_i \quad &+ \quad \sum_{k=1}^{3}\mathcal{Y}_i^{x_k}\mathbf{W}_i + \sum_{a_{ij}\in\mathcal{F}_d^i}\mathcal{Y}_{ij}\mathbf{W}_j \\
&+ \quad \sum_{a_{ij}\in\mathcal{F}_m^i}\mathcal{Y}_{im}\mathbf{W}_i + \sum_{a_{ij}\in\mathcal{F}_a^i}\mathcal{Y}_{ia}\mathbf{W}_i = -\sum_{a_{ij}\in\mathcal{F}_a^i}\mathcal{Y}_{i\infty}\mathbf{W}_i^{\infty}
\end{aligned}
$$

- $\mathbf{W}_i = {}^t(\mathbf{W}_{i1}, \mathbf{W}_{i2}, \cdots, \mathbf{W}_{id_i})$, $\mathbf{W}_i$ is a $6d_i \times 1$ vector
- $\mathcal{F}^i = \mathcal{F}_d^i \bigcup \mathcal{F}_a^i \bigcup \mathcal{F}_m^i$: set of faces of τ_i

Global algebraic system of equations: $A\mathbf{W} = b$

$$
\begin{cases}
A_{ii} &= \quad 2\mathbf{i}\omega\mathcal{Y}_i + \sum_{k=1}^{3}\mathcal{Y}_i^{x_k} + \sum_{a_{ij}\in\mathcal{F}_m^i}\mathcal{Y}_{im} + \sum_{a_{ij}\in\mathcal{F}_a^i}\mathcal{Y}_{ia} \\
A_{ij} &= \quad \mathcal{Y}_{ij}, \quad \text{for } j \in \mathcal{F}_d^i \quad , \quad b_i = \sum_{a_{ij}\in\mathcal{F}_a^i}\mathcal{Y}_{i\infty}\mathbf{W}_i^{\infty}
\end{cases}
$$

- A_{ii} and A_{ij} are $6d_i \times 6d_i$ matrices

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A_{ij} \quad &= \quad \mathcal{Y}_{ij}, \quad \text{for } j \in \mathcal{F}_d^i \quad , \quad b_i = \sum_{a_{ij}\in\mathcal{F}_a^i}\mathcal{Y}_{i\infty}\mathbf{W}_i^{\infty}
\end{aligned}
\right.
$$

- A_{ii} and A_{ij} are $6d_i \times 6d_i$ matrices

Discontinuous Galerkin discretization methods

Numerical results for the 2D time-harmonic Maxwell equations

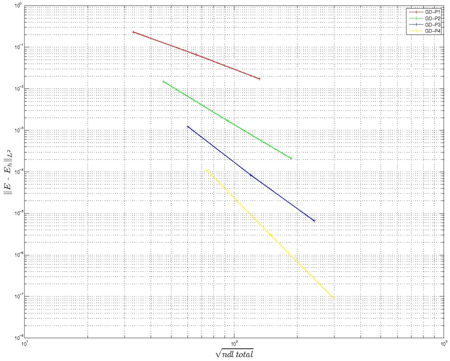
$$\left\{\begin{array}{l} \mu \mathbf{i}\omega H_x + \dfrac{\partial E_z}{\partial y} = 0 \\ \mu \mathbf{i}\omega H_y - \dfrac{\partial E_z}{\partial x} = 0 \\ \varepsilon \mathbf{i}\omega E_z - \dfrac{\partial H_y}{\partial x} + \dfrac{\partial H_x}{\partial y} = 0 \end{array}\right.$$

- DGTH-$\mathbb{P}_p$ method based on Lagrange (nodal) interpolation
  - Triangular mesh
  - Sparse block matrix, $3n_p \times 3n_p$ (with $n_p = ((p+1)(p+2))/2$)
  - MUMPS multifrontal sparse matrix solver
    (P.R. Amestoy, I.S. Duff and J.-Y. L'Excellent, CMAME, Vol. 184, 2000)

- Plane wave in vacuum, F=600 MHz
- Non-uniform triangular mesh
  - h_m and h_M: minimal and maximal edge length
  - CPU: factorization + solution time (Intel Xeon/2.33 GHz)

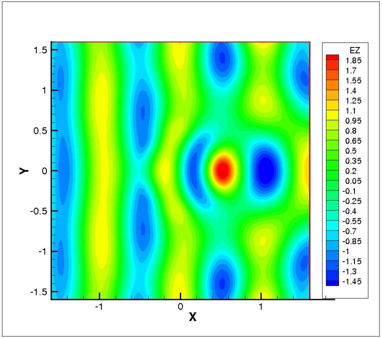
# triangles	n_z	h_m/h_M	Method	CPU	RAM (LU/total)
366	35,466	0.021/0.130	DG-$\mathbb{P}_1$	< 1.0 sec	1 MB/ 6 MB
-	108,120	-	DG-$\mathbb{P}_2$	< 1.0 sec	3 MB/ 16 MB
-	295,152	-	DG-$\mathbb{P}_3$	< 1.0 sec	6 MB/ 33 MB
-	644,922	-	DG-$\mathbb{P}_4$	< 2.0 sec	12 MB/ 60 MB
1,464	142,440	0.011/0.066	DG-$\mathbb{P}_1$	< 1.0 sec	5 MB/ 26 MB
-	433,200	-	DG-$\mathbb{P}_2$	< 1.0 sec	15 MB/ 69 MB
-	1,182,336	-	DG-$\mathbb{P}_3$	2.1 sec	32 MB/ 149 MB
-	2,582,712	-	DG-$\mathbb{P}_4$	4.0 sec	57 MB/ 266 MB
5,856	570,912	0.005/0.033	DG-$\mathbb{P}_1$	1.8 sec	28 MB/ 125 MB
-	1,734,240	-	DG-$\mathbb{P}_2$	4.7 sec	76 MB/ 326 MB
-	4,732,800	-	DG-$\mathbb{P}_3$	10.0 sec	156 MB/ 682 MB
-	10,336,896	-	DG-$\mathbb{P}_4$	20.6 sec	274 MB/1217 MB

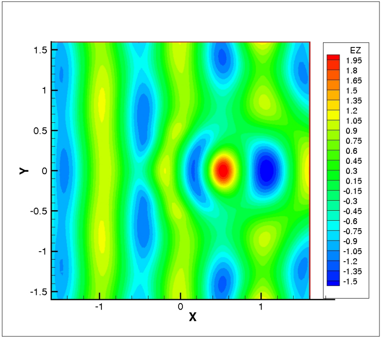
- Numerical convergence: non-uniform triangular mesh



	P1	P2	P3	P4
E_z	1.9	3.0	3.8	5.1
H	1.2	2.2	2.9	4.0

- Scattering of a plane wave by a dielectric cylinder, F=300 MHz





Contour lines of E_z
Left: exact solution - Right: DG-$\mathbb{P}_2$ method

- Scattering of a plane wave by a dielectric cylinder, F=300 MHz
  - # vertices = 4,108 and # elements = 8,054
  - h_m and h_M: minimal and maximal edge length
  - CPU: factorization + solution time (Intel Xeon/2.33 GHz)

n_z	h_m/h_M	Method	CPU	RAM (LU/total)
791,522	0.00071/0.14020	DG-$\mathbb{P}_1$	3.0 sec	50 MB/ 179 MB
2,402,104	-	DG-$\mathbb{P}_2$	13.4 sec	149 MB/ 463 MB
6,543,232	-	DG-$\mathbb{P}_3$	99.6 sec	366 MB/ 964 MB

Domain decomposition solver

Formulation in the continuous case

Time harmonic Maxwell system

$$\mathcal{L}\mathbf{W} = \mathrm{i}\omega G_0\mathbf{W} + G_x\partial_x\mathbf{W} + G_y\partial_y\mathbf{W} + G_z\partial_z\mathbf{W} = 0$$

- Flux matrices

$$G_l = \left[\begin{array}{cc} 0_{3\times 3} & N_l \\ -N_l & 0_{3\times 3} \end{array}\right] \quad \text{for} \quad l = x, y, z \quad \text{and with} \quad {}^tN_l = -N_l$$

- Property : for any $\mathbf{n} = {}^t(n_x, n_y, n_z)$ with $\parallel \mathbf{n} \parallel = 1$,

$$C(\mathbf{n}) = G_0^{-1}\left(n_x G_x + n_y G_y + n_z G_z\right) \quad \text{is diagonalizable}$$

$$C(\mathbf{n}) = T(\mathbf{n})\Lambda(\mathbf{n})T^{-1}(\mathbf{n})$$

$$\text{Eigenvalues} \ : \ \lambda_{1,2} = -c \ , \ \lambda_{3,4} = 0 \ , \ \lambda_{5,6} = c \text{ with } c = \frac{1}{\sqrt{\varepsilon\mu}}$$

Domain decomposition solver

Formulation in the continuous case

Schwarz algorithm

- $\Omega = \bigcup_{j=1}^{N_s} \Omega_j$, $\mathbf{W}^j = \mathbf{W}|_{\Omega_j}$
- $\Gamma = \Gamma_a$ (for the presentation)
- Overlapping subdomains

$$\left\{ \begin{array}{rcl} \mathcal{L}\mathbf{W}^{j,p+1} & = & 0 \text{ in } \Omega_j \\ \mathcal{B}_{\mathbf{n}_{jl}}\mathbf{W}^{j,p+1} & = & \mathcal{B}_{\mathbf{n}_{jl}}\mathbf{W}^{l,p} \text{ on } \Gamma_{jl} = \partial\Omega_j \cap \bar{\Omega}_l \\ G_{\mathbf{n}}^{-}\mathbf{W}^{j,p+1} & = & G_{\mathbf{n}}^{-}\mathbf{W}_{\mathrm{inc}} \text{ on } \Omega_j \cap \Gamma_a \end{array} \right.$$

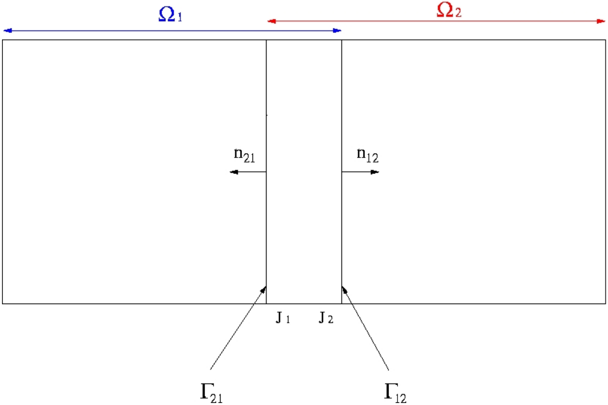
Classical (natural) interface conditions

$$\mathcal{B}_{\mathbf{n}} \equiv G_{\mathbf{n}}^{-}$$

$$G_{\mathbf{n}}^{-}\mathbf{W} \iff \mathbf{n} \times \mathbf{E} + z\mathbf{n} \times (\mathbf{n} \times \mathbf{H}) \text{ (impedance condition)}$$

- Two subdomain case and $\Omega = \mathbb{R}^2$

$$\Omega_1 =]-\infty, b[\times\mathbb{R} \quad \text{and} \quad \Omega_2 =]a, +\infty[\times\mathbb{R} \quad \text{with} \quad a \leq b$$



- V. Dolean, M.J. Gander and L. Gerardo-Giorda, SISC, to appear (2009)
- Fourier analysis

$$\widehat{\mathcal{E}}^{j,p}(x,k) = \mathcal{F}_y(\mathcal{E}^{j,p}) = \int_{\mathbb{R}} e^{-iky}\mathcal{E}^{j,p}(x,y)\,dy$$

with $\mathcal{E}^{j,p} = \mathbf{U}^{j,p} - \mathbf{U}^{ex}_{|\Omega_j}$ where $\mathbf{U} = T^{-1}\mathbf{W}$ (characteristic variables)

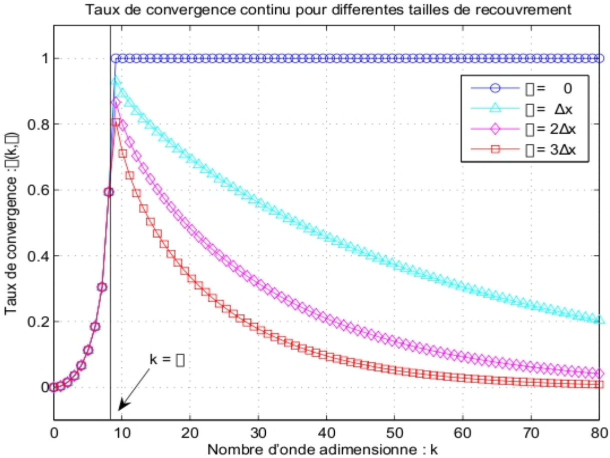
Convergence rate

$$\rho(k,\delta) = \left| \left(\frac{\sqrt{k^2-\omega^2} - \mathrm{i}\omega}{\sqrt{k^2-\omega^2} + \mathrm{i}\omega} \right) e^{-\delta\sqrt{k^2-\omega^2}} \right|$$

with $\delta = b - a$

$$\rho(k,\delta) = \begin{cases} \left| \dfrac{\sqrt{\omega^2-k^2} - \omega}{\sqrt{\omega^2-k^2} + \omega} \right| & \text{si } |k| < \omega \quad \text{(propagative modes)} \\ e^{-\delta\sqrt{k^2-\omega^2}} & \text{si } |k| \geq \omega \quad \text{(evanescent modes)} \end{cases}$$

Convergence rate as a function of the frequency parameter



Domain decomposition resolution algorithms

Schwarz algorithm: algebraic formulation

- Global system (two-sudomain case)

$$\begin{pmatrix} A_1 & 0 & R_1 & 0 \\ 0 & A_2 & 0 & R_2 \\ 0 & -B_2 & \mathrm{Id} & 0 \\ -B_1 & 0 & 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathbf{W}_h^1 \\ \mathbf{W}_h^2 \\ \lambda_h^1 \\ \lambda_h^2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h^1 \\ \mathbf{f}_h^2 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

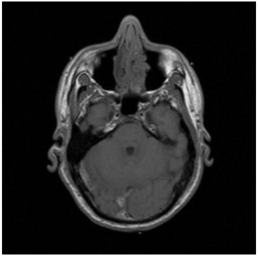
- Interface system: $\mathcal{T}_h \lambda_h = \mathbf{g}_h$

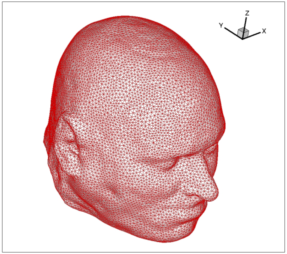
$$\mathcal{T}_h = \begin{pmatrix} \mathrm{Id} & B_2 A_2^{-1} R_2 \\ B_1 A_1^{-1} R_1 & \mathrm{Id} \end{pmatrix} \quad \text{and} \quad \mathbf{g}_h = \begin{pmatrix} B_2 A_2^{-1} F^2 \\ B_1 A_1^{-1} F^1 \end{pmatrix}$$

  - Schwarz iteration $\Leftrightarrow \lambda_h^{p+1} = (\mathrm{Id} - \mathcal{T}_h)\lambda_h^p + \mathbf{d}_h$
  - Accelerated iteration $\Rightarrow$ Krylov method

Domain decomposition resolution algorithms

3D application





Geometric models

- Built from segmented medical images
- Extraction of surfacic (triangular) meshes of the tissue interfaces using specific tools
  - Marching cubes + adaptive isotropic surface remeshing
  - Delaunay refinement
- Generation of tetrahedral meshes using a Delaunay/Voronoi tool

Domain decomposition resolution algorithms

3D application

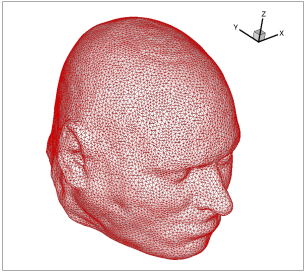
- Plane wave exposure: F=1.8 GHz
- Characteristics of the tetrahedral meshes

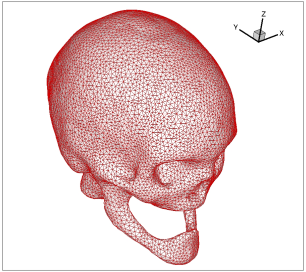
Mesh	# vertices	# tetrahedra	L_{min} (mm)	Lmax (mm)	Lavg (mm)
M1	188,101	1,118,952	9.04	23.86	9.09
M2	309,599	1,853,832	1.15	24.76	6.93

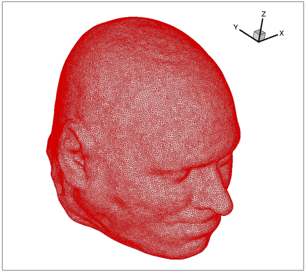
- Solution methods
  - Interface system
    - BiCGstab(ℓ) (G.L.G. Sleijpen and D.R. Fokkema, ETNA, Vol.1, 1993)
    - No preconditioner, $\ell = 6$
  - Local systems
    - Sparse direct solvers: MUMPS or PasTiX
    - Mixed arithmetic strategy: LU in 32 bit + iterative refinement
- Hardware platform
  - Bull Novascale 3045 system of the CEA/CCRT center (Centre de Calcul Recherche et Technologie)
  - Intel Itanium 2/1.6 GHz, InfiniBand

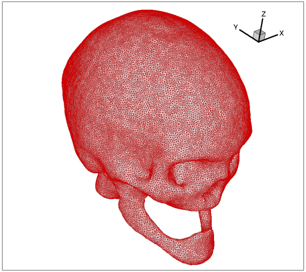
Domain decomposition resolution algorithms

3D application



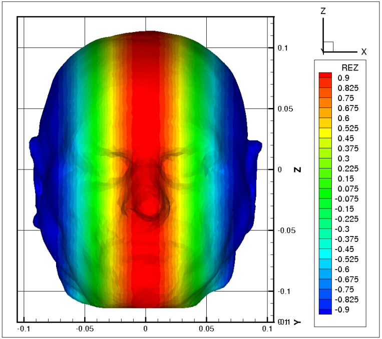


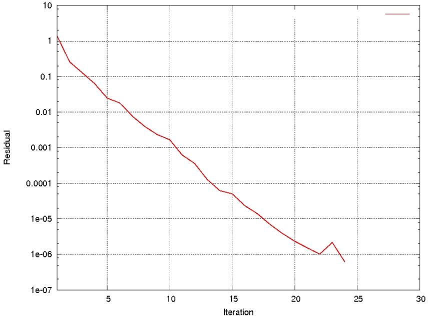




Domain decomposition resolution algorithms

3D application: homogeneous propagation media



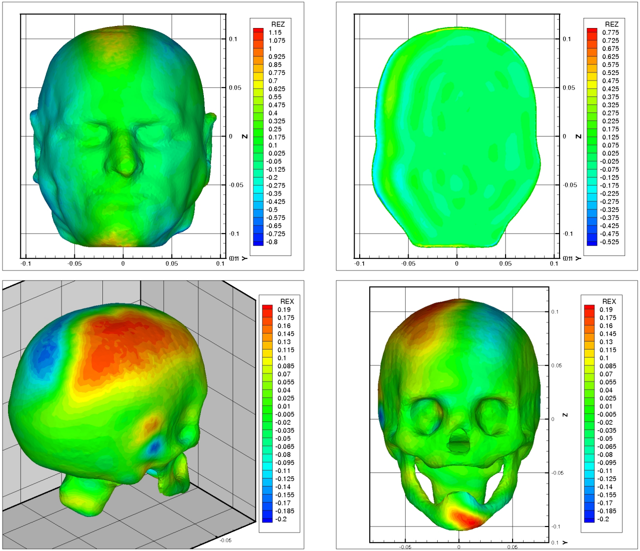


Mesh	Method	# d.o.f	N_s	# it	CPU (min/max)	Elapsed time
M1	DGTH-$\mathbb{P}_1$	26,854,848	160	24	1204 sec/1209 sec	1210 sec

LU (min/max)	CPU factor (min/max)	Elapsed time factor
2.1 GB/3.1 GB	493 sec/494 sec	495 sec

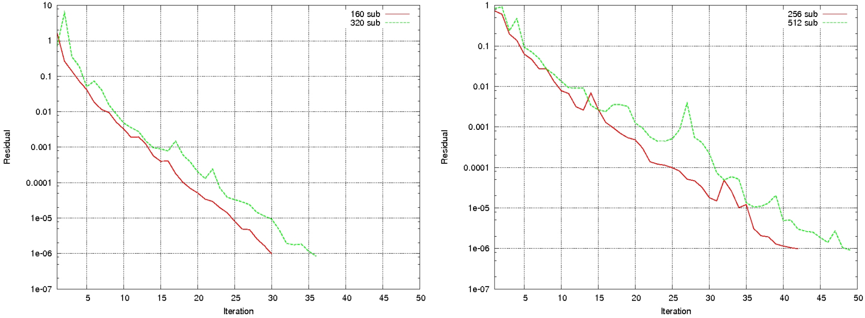
Domain decomposition resolution algorithms

3D application: heterogeneous propagation media



Domain decomposition resolution algorithms

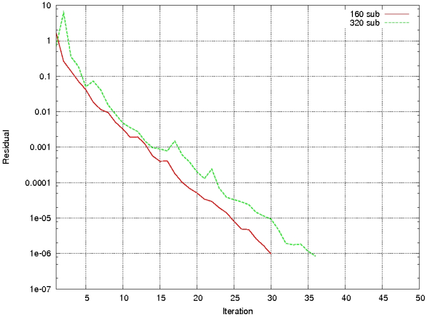
3D application: heterogeneous propagation media

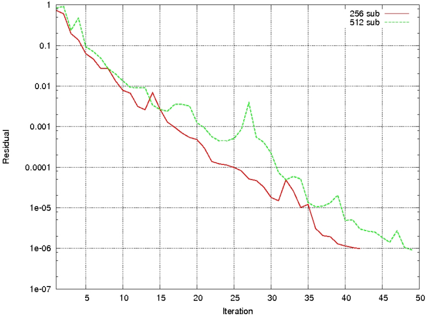


Mesh	Method	# d.o.f	N_s	# it	CPU (min/max)	Elapsed time
M1	DGTH-$\mathbb{P}_1$	26,854,848	160	30	1311 sec/1313 sec	1314 sec
-	-	-	320	36	525 sec/ 527 sec	528 sec (2.5)
M2	DGTH-$\mathbb{P}_1$	44,491,968	256	42	1816 sec/1823 sec	1824 sec
-	-	-	512	49	782 sec/ 784 sec	785 sec (2.3)

Domain decomposition resolution algorithms

3D application: heterogeneous propagation media





Mesh	N_s	LU (min/max)	CPU factor (min/max)	Elapsed time factor
M1	160	2.1 GB/3.1 GB	490 sec/495 sec	496 sec
-	320	0.8 GB/1.2 GB	130 sec/131 sec	132 sec (3.8)
M2	256	2.2 GB/3.2 GB	525 sec/527 sec	528 sec
-	512	0.8 GB/1.3 GB	138 sec/140 sec	142 sec (3.7)

DGTH method

- Non-conforming (both in h and p) DGTH method
- *hp*-adaptivity

Solution methods

- Schwarz algorithms based on optimized interface conditions (in collaboration with M. Gander, University of Geneva)
- Subdomain solver
  - Block ILU preconditioned iterative solver
  - Hierarchical solution strategies

High performance computing

- Hierarchical SPMD model (multiple parallelism levels)
- DG method on hybrid CPU/GPU systems

Closure

Thank you for your attention!